PRICING AND RISK MANAGEMENT OF VARIABLE ANNUITIES AND EQUITY-INDEXED ANNUITIES

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ABSTRACT

A variety of equity-linked insurance contracts such as variable annuities (VA) and equity-indexed annuities (EIA) have gained their attractiveness in the past decade because of the bullish equity market and low interest rates. Due to the complexity of their inherent nature, pricing and risk management of these products are quantitatively challenging and therefore have become sources of concern to many insurance companies. From a financial engineer's perspective, the options in VA and those embedded in EIA can be modeled as puts and calls, respectively, and enable the use of numerical option pricing techniques. Additionally, values of VA and EIA move in opposite directions in response to changes in the underlying equity value. Therefore, for insurers who offer both businesses, there are natural offsets or diversification benefits in terms of economic capital (EC) usage. In this chapter, we consider two specific products: the guaranteed minimal account benefit (GMAB) and the point-to-point (PTP) EIA contract, which belong to the VA and EIA classes respectively. Taking into account mortality risk and suboptimal dynamic lapse behavior, we build a framework that quantifies the value of each

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product and the natural hedging benefits based on risk-neutral option pricing theory. With Monte Carlo simulation and finite difference methods being implemented, an optimum product mixture of those two contracts is achieved that deploys capital the most efficiently.

1. INTRODUCTION

The market for equity-linked insurance such as variable annuities (VA) and equity-indexed annuities (EIA) has grown tremendously over recent years and has become a significant segment of our capital markets. This has been evidenced by the growing sales that have reached \$113 billion for VA and \$13 billion for EIA in 2003.[1] This is partly thanks to the bullish US equity market along with relatively low interest rates over the past decade, which have led policyholders to be more aware of investment opportunities outside the traditional insurance sector so that they can enjoy the benefits from financial markets in conjunction with investment guarantees and tax advantages. Different from traditional insurance products, these equity-linked insurance contracts provide policyholders mortality or maturity protection as well as the beneficial return based on the equity market's performance. The pricing and risk management of these products are quantitatively challenging and therefore have become sources of concern to both the regulator and the many insurance companies. For instance, pricing these annuity contracts is complicated with mortality risk and dynamic lapse[2] behavior involved; also, the limited capital of a life insurance company constrains the volume of its VA and EIA business; thus, how to deploy the *economic capital* (EC)[3] more efficiently turns out to be an urgent problem to frame.

It is important to stress that from an option pricing perspective, the options in VA and those embedded in EIA can be modeled as *puts* and *calls*, respectively, which will be shown in detail later. However, with mortality and dynamic lapse risk involved, pricing these contracts becomes numerically challenging and needs special techniques for its complicated features such as path dependency.

The values of these embedded options move in opposite directions in response to underlying equity price changes. Suppose both products share the same underlying equity process, then these two types of options have payoffs that can partially offset each other, thus natural diversification benefits exist in a portfolio that contains both VA and EIA products, and therefore, the EC requirements for that annuity writers can be reduced. From the insurance company's (risk management) point of view, it will be very useful to quantify these diversification benefits and derive an optimal business mix based on the most efficient way to deploy the capital. The framework of this chapter, which differs from previous literatures, is based on this purpose.

Perhaps, the best way to illustrate this intuition is through a simple numerical example. Table 1 provides the Value at Risk (VaR) and standard deviation of a European put, a European call, and a 50/50 mixture of these two options (i.e., a *straddle*) at time horizons of both 1 and 2 years. This example assumes both options are at-the-money, have maturity of 4 years, and are based on the same underlying equity price that follows a geometric Brownian motion with drift $\mu = 8\%$, non-dividend-paying, volatility $\sigma = 0.2$, risk-free rate $r = 2\%$, and initial price $S_0 = 1$.

It is shown in Table 1 that the straddle portfolio has a much lower VaR and standard deviation than the average of these two options, which can be explained by Fig. 1. The correlation between the prices of a put and a call is negative: when one option is in-the-money (implies a higher price), the other one is likely to be out-of-the-money (implies a lower price). This natural diversification lowers down both the VaR and the standard deviation of that straddle portfolio (red line in Fig. 1). And it will be shown later that similar diversification effect also exists in portfolio that contains both VA and EIA.

There have been some previous literature in this area. For research on VA, Brennan and Schwartz (1976), Boyle and Schwartz (1977), and Brennan and Schwartz (1979) first introduced the famous Black–Scholes–Merton (Merton, 1973) framework into this field. They assumed complete markets for both financial and mortality risk and derived risk-neutral price formulae. More recent work on equity-linked life insurance was done by Bacinello and Ortu (1993a, 1993b, 1996), Aase and Persson (1994), and Nielsen and

Table 1. Diversification of a Put and Call.

Tenor (Years)	99% Value at Risk			Standard Deviation		
	Put	Call	50/50 Mix	Put	Call	50/50 Mix
1	0.30	0.76	***0.38***	0.07	0.16	***0.06***
2	0.38	1.22	***0.61***	0.09	0.27	***0.11***

Note: The bold/italic numbers are used to show the effect of risk reduction (in terms of 99% VaR and Std. Dev.) from diversification.

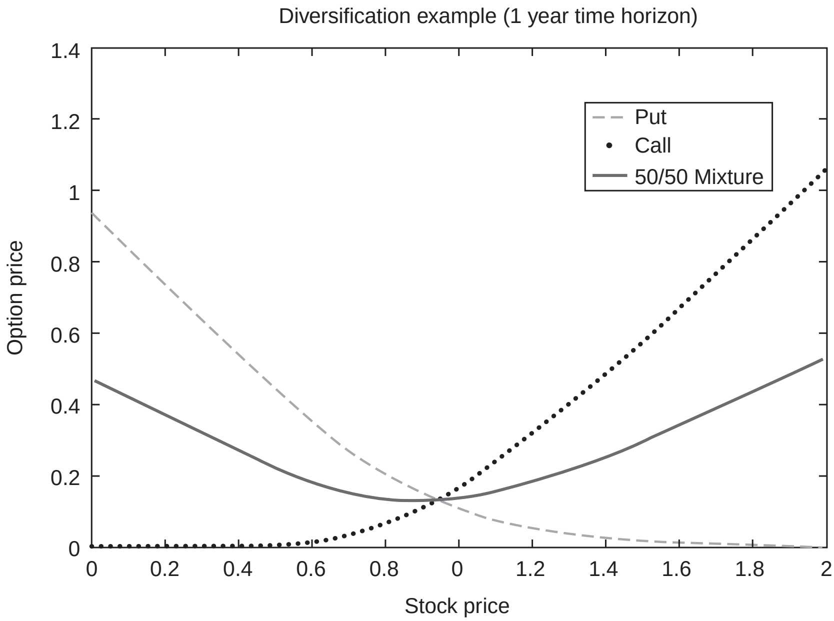


Fig. 1. Diversification of a Put and a Call. (1-year time horizon).

Sandmann (1995). These authors allowed the risk-free interest rate to be stochastic. Follmer and Sonderman (1986) assumed an incomplete mortality market and introduced the concept of risk-minimizing strategies, which was extended by Moller (1998). Hardy (2003) offered risk-neutral pricing and dynamic hedging analyses on VA. Milevsky applied an optimal control technique to analyze VA with mortality and lapse risk (Milevsky & Salisbury, 2002) as a best stopping time problem and concluded that in today's market, the guaranteed minimum death benefit (GMDB) products were overpriced (Milevsky & Posner, 2001), and in contrast, the guaranteed minimum withdrawal benefit (GMWB) products were underpriced (Milevsky & Salisbury, 2004).

In the field of EIA research, Tiong (2000) used Esscher transforms and derived closed form pricing formulae for several types of EIA products: point-to-point (PTP), cliquet, and lookback, which were also covered by Hardy (2003). Lin and Tan (2003) extended the model to include stochastic interest rates.

This chapter applied the Black–Scholes–Merton option pricing framework along with a complete mortality and lapse market. Oppose to an *optimal control* approach, lapse behavior is modeled as a function of time and underlying equity performance that can be economically irrational and suboptimal. Based on this framework, we developed analytic formulas and finite difference schemes to price both VA and EIA, which enable the EC calculation and optimization.

The rest of this chapter is organized as follows. We present the framework in Section 2. Analytical formulas including risk-neutral pricing and EC calculating are implemented on two specific products: guaranteed minimum account benefit (GMAB) in Section 2.1, and the PTP EIA contract in Section 2.2, which belong to the VA and EIA classes respectively. In Section 2.3, we introduced a finite difference approach to price GMAB and PTP. In Section 2.4, we analyzed the EC of a GMAB/PTP mixture portfolio based on a Monte Carlo simulation and finite difference hybrid algorithm. An optimal combination of these two products is achieved which employs EC the most efficiently. We conclude in Section 2.5 with closing remarks and summary.

2. FORMULATION

2.1. GMAB Contract, Valuation, and Economic Capital

2.1.1. Product Description

VA are tax-deferred, complex structured equity, and interest rate investment vehicles. They provide money-back guarantees on a separate mutual fund account, and these guarantees can be viewed as put options with an increasing strike price. Different from usual financial products that are paid up-front, premiums of these products are paid by installments, with a proportional benefit charge that is deduced from the underlying mutual fund account on a periodic basis.

The simplest VA product is the GMAB, which provides the beneficiary a minimal guarantee in the event that the policyholder dies or contract matures, whichever one comes first. In this chapter, we focus on a GMAB account.

An example of a GMAB contract is as follows: at initiation, $t = 0$, the policyholder enters into a contract by paying the insurance company an initial amount P. The insurance company immediately invests the amount P into a mutual fund (such as an S&P 500 index fund) and there is no further payment from the policyholder. The insurance company guarantees a rate of return r_g up to the end of contract, when the beneficiary will receive the greater of either the current mutual fund account value or the guaranteed amount. In exchange, the insurance company charges a certain percent of account amount as the contract fees. The guaranteed payment can be triggered by mortality or maturity, but not by lapse behavior: If the policyholder decides to lapse the VA contract before maturity, he/she can get his/her mutual fund account value back after some penalty fees charged, but the guarantee is not redeemable.

2.1.2. GMAB without Mortality and Lapse

Consider a GMAB contract with \$1 initial account value and maturity time N (in years). Ignoring any mortality and lapse risk, the embedded option in GMAB turns to be a plain vanilla European put.

For the rest of this chapter, the underlying equity price is assumed to satisfy a geometric Brownian motion, the interest rate is assumed to be constant, and continuous compounding will be used for simplicity. This framework is similar to Hardy (2003). Given time horizon n prior to maturity, let G_n be the guaranteed amount,

$$G_n = \mathrm{e}^{r_g n} \cdot 1, \quad 0 \leq n \leq N$$

As we discussed before, G_n is going to be the strike price for its embedded option. Let m be management fee rate that was charged to policyholder's account and $\{F_n\}$ be the account value process that satisfies,

$$F_n = \mathrm{e}^{-mn} \frac{S_n}{S_0}, \quad 0 \leq n \leq N$$

At any time $t = n$ prior to N, suppose the underlying stock price is S_n. The embedded put option value in GMAB can then be calculated as follows:

$$H_n = E_n^Q[\mathrm{e}^{-r(N-n)} H_N]$$

where

$$H_N = (G_N - F_N)^+ = \left(\mathrm{e}^{r_g N} - \mathrm{e}^{-mN} \frac{S_N}{S_0}\right)^+$$
$$= \frac{\mathrm{e}^{-mN}}{S_0} \left(\mathrm{e}^{(m+r_g)N} S_0 - S_N\right)^+$$

In the formula above, $E_n^Q[\bullet]$ is expectation under risk-neutral measure Q. The H_N term, which is the final cash flow of the GMAB contract that happens at maturity N, is equivalent to the payoff of a vanilla European put option. If we take notation $V_{\mathrm{put}}(S_0, K, r, d, \sigma, t)$ as the price of a vanilla European put, then under the Black–Scholes–Merton framework (Black & Scholes, 1973), the closed form of such an option value can be written as follows:

$$\begin{aligned} H_n &= \frac{\mathrm{e}^{-mN}}{S_0} \cdot V_{\mathrm{put}}(S_n, \mathrm{e}^{(m+r_g)N} S_0, r, d, \sigma, N-n) \\ &= \mathrm{e}^{r_g N - r(N-n)} \Phi(-d_2) - \frac{\mathrm{e}^{-mN}}{S_0} S_n \mathrm{e}^{-d(N-n)} \Phi(-d_1) \end{aligned}$$

where

$$\begin{aligned} d_1 &= \frac{\log(S_n/S_0) - (m + r_g)N + (r - d + \frac{\sigma^2}{2})(N-n)}{\sigma\sqrt{N-n}} \\ d_2 &= d_1 - \sigma\sqrt{N-n} \end{aligned}$$

For a GMAB contract, the net value of adding the guarantee to the VA product at time n, noted by $\mathrm{NV}_n(S_n)$, can be formulated as the difference between two parts: the embedded option (guarantee) value from time n to maturity N, and the present value of the benefit charge (noted as f_n), as a portion of the total management fees charged to the policyholder's account. $\mathrm{NV}_n(S_n)$ has the following form:

$$\mathrm{NV}_n(S_n) = H_n - f_n$$

and

$$f_n = \int_n^N \mathrm{e}^{-r(t-n)} E_n^Q[F_t] m\,dt = \frac{1}{S_0} S_n \int_n^N \mathrm{e}^{-mt} m\,dt = \frac{1}{S_0} S_n [\mathrm{e}^{-mn} - \mathrm{e}^{-mN}]$$

The corresponding EC of GMAB is defined as the percentile risk measure of $\mathrm{NV}(S_n)$:

$$\mathrm{P}[\mathrm{NV}_n(S_n) - \mathrm{NV}_0(S_0) \geq \mathrm{EC}_{\mathrm{GMAB}}] < 1 - \beta$$

where β is the confidence level. As $\mathrm{NV}_n(S_n)$ is monotonic,[4] its analytical EC (or equivalent, VaR) can be directly calculated (Fong & Lin, 1999) in the following way:

$$\mathrm{Var}[f(S)] = f(\mathrm{Var}[S]) \quad \text{if } f(S) \text{ is monotonic}$$

Supposing that a 99% confidence level (notice this is under realistic measure) is applied, the EC under current framework is as follows:

$$\begin{aligned}
\mathrm{EC_{GMAB}} &= \mathrm{NV}_{n,99\%} - \mathrm{NV}_0 = H_{n,99\%} - f_{n,99\%} - \mathrm{NV}_0 \\
&= \frac{\mathrm{e}^{-mN}}{S_0} \cdot \mathrm{V_{put}}(S_{n,99\%}, \mathrm{e}^{(m+r_g)N} S_0, r, d, \sigma, N-n) - f_{n,99\%} - \mathrm{NV}_0 \\
&= \mathrm{e}^{r_g N}\Phi(-d_2) - \mathrm{e}^{-mN-d(N-n)}\mathrm{e}^{(\mu-d-\frac{\sigma^2}{2})n-2.33\sigma\sqrt{n}}\Phi(-d_1) \\
&\quad - f_n(S_0 e^{(\mu-d-\frac{\sigma^2}{2})n-2.33\sigma\sqrt{n}}) - \mathrm{NV}_0
\end{aligned}$$

where

$$\begin{aligned}
d_1 &= \frac{(\mu-d-\frac{\sigma^2}{2})n - 2.33\sigma\sqrt{n} - (m+r_g)N + (r-d+\frac{\sigma^2}{2})(N-n)}{\sigma\sqrt{N-n}} \\
d_2 &= d_1 - \sigma\sqrt{N-n} \\
\mathrm{NV}_0 &= H_0 - f_0
\end{aligned}$$

2.1.3. GMAB with Mortality and Lapse

In the previous section, mortality and lapse risk were totally ignored. In the real world, it is the involvement of mortality and lapse that distinguish GMAB from the normal financial instruments. Mortality leads to stochastic contract maturity time, and the lapse feature gives the policyholder an opportunity to abandon the contract. (Lapse happens when policyholders terminate payments without having paid the full value of contract, usually at the cost of penalty.)

Let $\Psi(t)$ be the percentage of policyholders that survive and do not lapse before time t, $q(t)$ and $l(S_t, t)$ be the simultaneous mortality and lapse intensities (or *hazard rates*), respectively. Independence between lapse risk and mortality risk is also assumed. Under a continuous time model, $\Psi(t)$ has the following form:

$$\Psi(t) = \mathrm{e}^{-\int_0^t [l(S_u,u)+q(u)]du}$$

Standard actuarial practice treats mortality risk as *diversifiable* or *non-systematic*, which means the mortality risk can be eliminated by issuing a large enough number of equivalent contracts.[5] In this chapter, we adhere to this assumption. Then, the benefits of a life insurance contract turn to be $\int \Psi(t)q(t)P(t)\,dt$, where $P(t)$ represents the payoff at time t.

However, because equity market performance has huge impact on the policyholder's lapse behavior (Shumrak, Greenbaum, Darley, & Axtell, 1999; Milevsky & Salisbury, 2002), lapse risk is not fully diversifiable and therefore cannot be hedged by simply issuing a large number of contracts. Lapse rate l has form of $l(S_t, t)$, and survival probability $\Psi(t)$ depends on the whole underlying equity price path $\{S_t\}$ prior to n.

A number of researchers model the lapse behavior as a policyholder's rational decision and treat VA as an American-typed option with best stopping time always approachable. In this chapter, we suggest that the lapse behavior of both VA and EIA policyholders can be irrational and suboptimal just like other life insurance products and build the model in a different way.[6]

We introduce the *dynamic lapse multiplier* to model dynamic lapse. At any time n, the instantaneous lapse rate can be modeled as follows:

$$l(S_n, t) = f(R, t) \cdot l_B$$

where

$$R = \frac{F_n}{G_n} = \frac{1}{S_0} S_n \mathrm{e}^{-(m+r_g)n}$$

The actual lapse rate l is the product of the base lapse rate l_B[7] and the dynamic lapse multiplier $f(R, t)$. $f(R, t)$ depends on the ratio of guaranteed value to market value (GV/MV). The dynamic lapse multiplier is a non-decreasing function in variable S_n, which means a GMAB policyholder is more likely to lapse when the embedded option is more out-of-the-money (i.e., when the ratio of account value and guarantee is high).

Taking survival probability into account, the risk-neutral price of the embedded option is as follows:

$$H_n = E_n^Q\left[\int_n^N \mathrm{e}^{-r(t-n)}\Psi(t)q(t)(G_t - F_t)^+ dt + \mathrm{e}^{-r(N-n)}\Psi(N)\cdot(G_N - F_N)^+\right] \tag{1}$$

and the PV of the fees

$$f_n = E_n^Q\left[\int_n^N \Psi(t)\cdot \mathrm{e}^{-r(t-n)} F_t m dt\right]$$

Let $\mathrm{NV}_n(S_n)$ be the net value of adding the guarantee to the VA product, which is

$$\mathrm{NV}_n(S_n) = H_n - f_n$$

Taking into account the mortality and lapse risk, EC of GMAB is defined in the same way as previously defined,

$$P[(\mathrm{NV}_n(S_n) - \mathrm{NV}_0(S_0)) \geq \mathrm{EC}_{\mathrm{GMAB}}] < 1 - \beta$$

Because of the path dependency of $NV_n(S_n)$, an analytic form of $\mathrm{EC}_{\mathrm{GMAB}}$ is difficult to achieve. In later sections of this chapter, a Monte Carlo simulation/finite difference hybrid algorithm is implemented to calculate $\mathrm{EC}_{\mathrm{GMAB}}$.

Fig. 2 illustrates the impact of mortality and lapse risk on the VA embedded option value H_n and fee amount f_n as function of underlying equity price S, shown as red and blue lines respectively. The solid lines are the case when no mortality and lapse risk is taken into account; the dashed lines in the left plot represent a 5% constant mortality rate model, and the dashed lines in the right plot denote a 5% base lapse rate model. As observed, mortality risk reduces fee amount f_n (as less people pay premium payments) and has opposite effect on H_n upon underlying equity price.

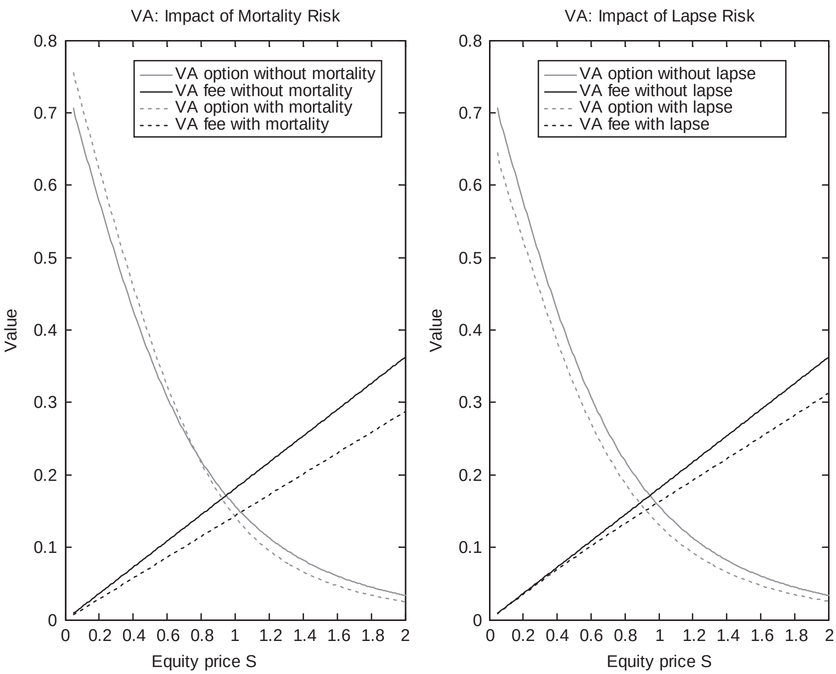


Fig. 2. Impact of Mortality and Lapse Risk on VA.

Mortality triggers early exercise; thus, when equity price is low, mortality causes in-the-money put option exercise and the contract becomes less profitable to the insurer; when equity price is high, early exercise is suboptimal to policyholders as the put option time value is lost entirely, turns the contract to be more in insurer's favor. In contrast, lapse behavior always reduces fee amount f_n and H_n, as less people keep paying premium and more people surrender their embedded options.

In a simpler case, if lapse risks are assumed to be independent from the market (implying $l(t)$ does not depend on S_n), a more simplified form of the GMAB would be accessible. Let $BSP(n,t)$ be, at any time n, the value of the put option embedded in GMAB that matures at t, without taking lapse and mortality into account. From the previous section we know that

$$\begin{aligned}\mathrm{BSP}(n,t) &= \frac{\mathrm{e}^{-mt}}{S_0}\cdot V_{\mathrm{put}}(S_n, \mathrm{e}^{(m+r_g)t}S_0, r, d, \sigma, t-n)\\ &= \mathrm{e}^{r_g t - r(t-n)}\Phi(-d_2) - \frac{\mathrm{e}^{-mt}}{S_0}S_n\mathrm{e}^{-d(t-n)}\Phi(-d_1)\end{aligned}$$

where

$$\begin{aligned}d_1 &= \frac{\log(S_n/S_0) - (m+r_g)t + (r - d + \frac{\sigma^2}{2})(t-n)}{\sigma\sqrt{t-n}}\\ d_2 &= d_1 - \sigma\sqrt{t-n}\end{aligned}$$

Unlike Eq. (1), $\Psi(t)$ is no longer path-dependent and therefore can be factored out from the risk-neutral expectation. The embedded put option value in GMAB can be written as follows:

$$H_n = \int_n^N \Psi(t)q(t)\mathrm{BSP}(n,t)dt + \Psi(N)\cdot\mathrm{BSP}(n,N)$$

The PV of the fees is

$$\begin{aligned}f_n &= \int_n^N \mathrm{e}^{-\int_0^t [l(u)+q(u)]du}\cdot \mathrm{e}^{-r(t-n)}E^Q[F_t]m\,dt\\ &= \frac{mS_n}{S_0}\int_n^N \mathrm{e}^{-\int_0^t [l(u)+q(u)]du}\cdot \mathrm{e}^{-m(t-n)}dt\end{aligned}$$

Proposition 1. In the case where both mortality and lapse risk are independent from the underlying equity prices, function $\mathrm{NV}_n(S_n)$ is monotonically decreasing.

Proof. See the appendix. ■

As $NV_n(S_n)$ is monotonic, its analytical EC (or equivalent, VaR) can be directly calculated in the same way as in the previous section (Fong & Lin, 1999):

$$EC_{GMAB} = NV_{n,99\%} - NV_0 = H_{n,99\%} - f_{n,99\%} - NV_0$$

where

$$H_{n,99\%} = \int_n^N \Psi(t)q(t)BSP_{99\%}(n,t)dt + \Psi(N)BSP_{99\%}(n,N)$$

$$f_{n,99\%} = e^{(\mu-d-\frac{\sigma^2}{2})n-2.33\sigma\sqrt{n}} m \int_n^N e^{-\int_0^t [l(u)+q(u)]du} \cdot e^{-m(t-n)} dt$$

$$\begin{aligned} BSP_{99\%}(n,t) &= \frac{e^{-mt}}{S_0} \cdot V_{put}(S_0 e^{(\mu-d-\frac{\sigma^2}{2})n-2.33\sigma\sqrt{n}}, e^{(m+r_g)t}S_0, r, d, \sigma, t-n) \\ &= e^{r_g t - r(t-n)}\Phi(-d_2) - e^{-mt}e^{(\mu-d-\frac{\sigma^2}{2})n-2.33\sigma\sqrt{n}}e^{-d(t-n)}\Phi(-d_1) \end{aligned}$$

$$\begin{aligned} d_1 &= \frac{(\mu-d-\frac{\sigma^2}{2})n - 2.33\sigma\sqrt{n} - (m+r_g)t + (r-d+\frac{\sigma^2}{2})(t-n)}{\sigma\sqrt{t-n}} \\ d_2 &= d_1 - \sigma\sqrt{t-n} \\ NV_0 &= H_0 - f_0 \end{aligned}$$

2.2. Valuation and Economic Capital of PTP

2.2.1. Product Description

Unlike VA, EIA are general account[8] assets. EIA contracts vary between insurance companies and the simplest EIA product is called PTP. This provides the beneficiary return on an index, but with a minimal guarantee (which is call-like) at the contract's maturity (usually death protection is included).

An example of a PTP contract is as follows: at the initiation, $t = 0$, the policyholder enters into a contract by paying the insurance company an initial amount P. The insurance company invests the amount P into the bond market, and there is no further payment from the policyholder.

The insurance company guarantees a fixed rate of return r_g (with a pre-specified guaranteed proportion) up to the end of the contract (guaranteed payment can be caused by mortality, maturity, or lapse decided by the policyholder), when the beneficiary will receive the greater of either the return on an index (with a pre-specified participation rate) or the guaranteed amount. If the policyholder lapses the EIA contract before maturity, he/she can get the guaranteed amount back after some penalty fees are charged, but the return on that index is not redeemable.

2.2.2. PTP without Mortality and Lapse

Consider a simple PTP contract with \$1 initial account value and maturity time N (in years) with fixed-interest rate r_g and guaranteed proportion η (95% or 100% is common). Also, assume the underlying equity index price follows geometric Brownian motion with constant risk-free rate and volatility. Let

$$G_n = \eta \cdot \mathrm{e}^{r_g n}, \quad 0 \le n \le N$$

be the amount of account value that is guaranteed. Similar to a GMAB contract, G_n is going to be the strike price for its embedded option. Let S_n represent the value at n of the equity index used. Given a participation rate α, the beneficiary of embedded call option payoff at maturity will be

$$\begin{aligned} H_N = (F_N - G_n)^+ &= \left(\left(1 + \alpha \left(\frac{S_N}{S_0} - 1 \right) \right) - \eta \cdot \mathrm{e}^{r_g N} \right)^+ \\ &= \frac{\alpha}{S_0} \left[S_N - \frac{S_0}{\alpha} (\eta \mathrm{e}^{r_g N} - (1 - \alpha)) \right]^+ \end{aligned}$$

with

$$F_N = \left(1 + \alpha \left(\frac{S_N}{S_0} - 1 \right) \right)$$

where F_N is the available amount for participation. At any time $n < N$, the embedded call value on this contract can be formulated through risk-neutral pricing theory.

$$H_n = E_n^Q[\mathrm{e}^{-r(N-n)} H_N]$$

Let notation $V_{\text{call}}(S_0, K, r, d, \sigma, t)$ represent the price of a standard European call. Under the Black–Scholes–Merton framework (Black & Scholes, 1973), the closed form of the embedded option value H_n can be written as follows:

$$H_n = \frac{\alpha}{S_0} \cdot V_{\text{call}}\left(S_n, \frac{S_0}{\alpha}(\eta e^{r_g N} - (1-\alpha)), r, d, \sigma, N-n\right)$$
$$= e^{-d(N-n)} \frac{\alpha S_n}{S_0} \Phi(d_1) - (\eta e^{r_g N} - (1-\alpha)) e^{-r(N-n)} \Phi(d_2)$$

where

$$d_1 = \frac{\log(\alpha S_n / [S_0(\eta e^{r_g N} - (1-\alpha))]) + \left(r - d + \frac{\sigma^2}{2}\right)(N-n)}{\sigma\sqrt{N-n}}$$
$$d_2 = d_1 - \sigma\sqrt{N-n}$$

Similar to a GMAB contract, the net value of adding the guarantee to the PTP product at time n, noted by $\mathrm{NV}_n(S_n)$, can be formulated as the difference between two parts: the first part is the embedded option (guarantee) value, from time n to maturity N; the second part is the present value of the fee that is going to be charged in the future (noted as f_n). $\mathrm{NV}_n(S_n)$ has the following form:

$$\mathrm{NV}_n(S_n) = H_n - f_n$$

where

$$f_n = \int_n^N e^{-r(t-n)}(r - r_g)\eta dt = \frac{r - r_g}{r}\eta[1 - e^{-r(N-n)}]$$

The corresponding EC of the PTP is defined as the percentile risk measure of $\mathrm{NV}(S_n)$:

$$P[\mathrm{NV}_n(S_n) - \mathrm{NV}_0(S_0) \geq \mathrm{EC}_{\mathrm{PTP}}] < 1 - \beta$$

where β is the confidence level. As $\mathrm{NV}_n(S_n)$ is again monotonic,[9] its analytical EC (or equivalent, VaR) is accessible (Fong & Lin, 1999). Supposing that a 99% confidence level (notice this is under realistic measure) is applied, the EC under the current framework is as follows:

$$\mathrm{EC}_{\mathrm{PTP}} = \mathrm{NV}_{n,99\%} - \mathrm{NV}_0 = H_{n,99\%} - f_n - \mathrm{NV}_0$$
$$= e^{-d(N-n)} \alpha e^{(\mu - d - \frac{\sigma^2}{2})n + 2.33\sigma\sqrt{n}} \Phi(d_1)$$
$$- (\eta e^{r_g N} - (1-\alpha)) e^{-r(N-n)} \Phi(d_2) - f_n - \mathrm{NV}_0$$

with

$$d_1 = \frac{\left(\mu - d - \frac{\sigma^2}{2}\right)n + 2.33\sigma\sqrt{n} + \log(\alpha/[(\eta e^{r_g N} - (1-\alpha))]) + \left(r - d + \frac{\sigma^2}{2}\right)(N-n)}{\sigma\sqrt{N-n}}$$

$$d_2 = d_1 - \sigma\sqrt{N-n}$$

$$\mathrm{NV}_0 = H_0 - f_0$$

2.2.3. PTP with Mortality and Lapse

The effect of taking mortality into consideration in a PTP contract is similar to the GMAB case. By using the same terminology, let $\Psi(t)$ be the percentage of policyholders that survive and do not lapse before t, $q(t)$ and $l(S_t, t)$ be the mortality and lapse intensities (or equivalently, *hazard rates*), respectively. Independence between lapse risk and mortality risk is also assumed. Then we see that

$$\Psi(t) = e^{-\int_0^t [l(S_u,u)+q(u)]du}$$

Similar to GMAB, lapse risk is not fully diversifiable and $\Psi(t)$ depends on the whole underlying equity price path $\{S_n\}$ prior to t. At any time n, the instantaneous lapse rate can be modeled as follows:

$$l(S_n, t) = f(R, t) \cdot l_B$$

with

$$R = \frac{G_n}{F_n} = \frac{S_0}{S_n}\eta \cdot e^{r_g n}$$

The actual lapse rate l is the product of the base lapse rate l_B and the dynamic lapse multiplier $f(R, t)$. $f(R, t)$ depends on the ratio of market value to guaranteed value (MV/GV, which is different from GMAB). The dynamic lapse multiplier is again a non-decreasing function in variable S_n, which means a PTP policyholder tends to lapse more likely when the embedded option is more out-of-the-money (i.e., when the ratio of account value and guarantee is high).

Taking survival probability into account, the risk-neutral price of the embedded option at time n is as follows:

$$H_n = E_n^Q\left[\int_n^N e^{-r(t-n)}\Psi(t)q(t)(F_t - G_t)^+ dt + e^{-r(N-n)}\Psi(N)\cdot(F_N - G_N)^+\right]$$

And the PV of the fees is

$$f_n = \int_n^N \Psi(t) \cdot e^{-r(t-n)}(r - r_g)\eta dt$$
$$= (r - r_g)\eta \int_n^N \Psi(t) \cdot e^{-r(t-n)} dt$$

Let $NV_n(S_n)$ be the net value of adding the guarantee to the PTP product, which is

$$NV_n(S_n) = H_n - f_n$$

Taking into account the mortality and lapse risk, the EC of PTP is defined as follows:

$$P[NV_n(S_n) - NV_0(S_0) \geq EC_{PTP}] < 1 - \beta$$

An analytical form of EC_{PTP} is difficult to achieve. In this chapter, a Monte Carlo simulation/finite difference hybrid algorithm is implemented to calculate EC_{PTP}.

Fig. 3 illustrates the impact of mortality and lapse risk on the EIA embedded option value H_n and fee amount f_n as function of underlying equity price S, shown as red and blue lines respectively. The solid lines are the case when no mortality and lapse risk is taken into account; the dashed lines in the left plot represent a 5% constant mortality rate model, and the dashed lines in the right plot denote a 5% base lapse rate model. Similar to Fig. 2, mortality risk reduces fee amount f_n (as less people pay premium payments) and has opposite effect on H_n upon underlying equity price. Mortality triggers early exercise; thus, when equity price is low, the out-the-money call option is exercised unprofitably and the contract becomes more profitable to the insurer; when equity price is high, call option is in-the-money and early exercise is in policyholder's favor. In contrast, lapse behavior always reduces fee amount f_n and H_n, as less people keep paying premium and more people surrender their embedded options.

In a simpler case, if lapse risks are assumed to be independent from the market (which means $l(t)$ does not depend on S_n), a clearer form of the PTP would be accessible. Let $BSC(n, t)$ be, at any time n, the value of the call option embedded in PTP that matures at t, without taking lapse and

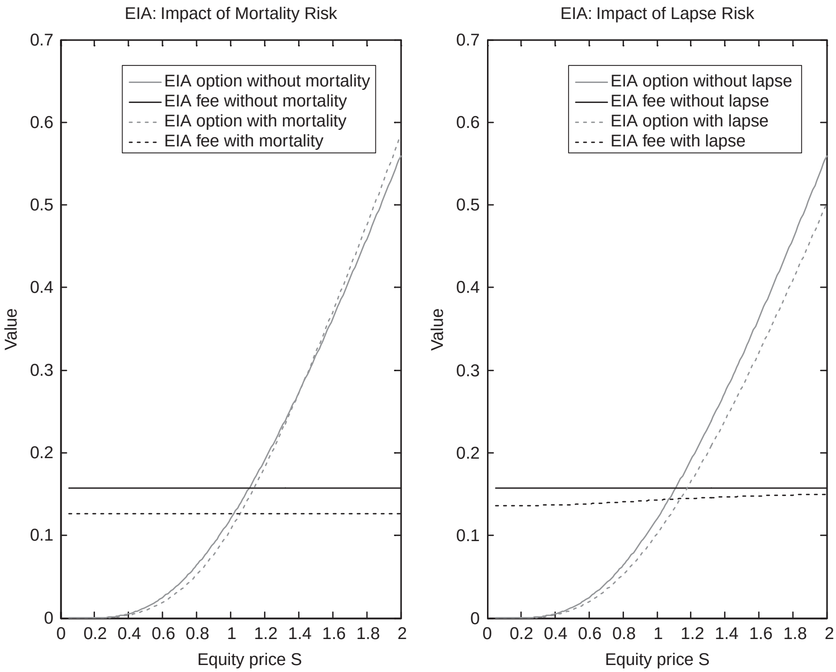


Fig. 3. Impact of Mortality and Lapse Risk on EIA.

mortality into account. From the previous section, we know that

$$\begin{aligned}\mathrm{BSC}(n,t) &= \frac{\alpha}{S_0}\cdot V_{\mathrm{call}}\left(S_n, \frac{S_0}{\alpha}(\eta \mathrm{e}^{r_g t} - (1-\alpha)), r, d, \sigma, t-n\right)\\ &= \mathrm{e}^{-d(t-n)}\frac{\alpha S_n}{S_0}\Phi(d_1) - (\eta \mathrm{e}^{r_g t} - (1-\alpha))\mathrm{e}^{-r(t-n)}\Phi(d_2)\end{aligned}$$

with

$$d_1 = \frac{\log(\alpha S_n/[S_0(\eta \mathrm{e}^{r_g t} - (1-\alpha))]) + \left(r - d + \frac{\sigma^2}{2}\right)(t-n)}{\sigma\sqrt{t-n}}$$

$$d_2 = d_1 - \sigma\sqrt{t-n}$$

Here $\Psi(t)$ is no longer path-dependent and therefore can be factored out from the risk-neutral expectation. The embedded call option value in PTP can be written as follows:

$$H_n = \int_n^N \Psi(t)q(t)\mathrm{BSC}(n,t)dt + \Psi(N)\cdot \mathrm{BSC}(n,N)$$

The PV of the fees is

$$f_n = (r - r_g)\eta \int_n^N \Psi(t)\cdot \mathrm{e}^{-r(t-n)}dt$$

where $\mathrm{NV}_n(S_n)$ is again monotonic through similar steps to those in the proof of Proposition 1. The EC of PTP can be calculated through the same way as in the last section (Fong & Lin, 1999):

$$\mathrm{EC}_{\mathrm{PTP}} = \mathrm{NV}_{n,99\%} - \mathrm{NV}_0 = H_{n,99\%} - f_n - \mathrm{NV}_0$$

with

$$H_n = \int_n^N \Psi(t)q(t)\mathrm{BSC}_{99\%}(n,t)dt + \Psi(N)\cdot \mathrm{BSC}_{99\%}(n,N)$$

$$\begin{aligned}\mathrm{BSC}_{99\%}(n,t) &= \frac{\alpha}{S_0}\cdot V_{\mathrm{call}}\left(S_0\mathrm{e}^{(\mu-d-\frac{\sigma^2}{2})n+2.33\sigma\sqrt{n}}, \frac{S_0}{\alpha}(\eta\mathrm{e}^{r_g t} - (1-\alpha)), r, d, \sigma, t-n\right)\\ &= \mathrm{e}^{-d(t-n)}\alpha\mathrm{e}^{(\mu-d-\frac{\sigma^2}{2})n+2.33\sigma\sqrt{n}}\Phi(d_1) - (\eta\mathrm{e}^{r_g t} - (1-\alpha))\mathrm{e}^{-r(t-n)}\Phi(d_2)\end{aligned}$$

$$d_1 = \frac{\left(\mu - d - \frac{\sigma^2}{2}\right)n + 2.33\sigma\sqrt{n} + \log(\alpha/[\eta\mathrm{e}^{r_g t} - (1-\alpha)]) + \left(r - d + \frac{\sigma^2}{2}\right)(t-n)}{\sigma\sqrt{t-n}}$$

$$d_2 = d_1 - \sigma\sqrt{t-n}$$

$$\mathrm{NV}_0 = H_0 - f_0$$

2.3. *Pricing VA/EIA: A Finite Difference Approach*

2.3.1. *Methodology*

Oppose to the Monte Carlo simulation, finite difference is a fast and highly efficient method to compute the irregular and path-dependent VA/EIA option value.[10]

The first step we take is to remove path dependency. Let $H_n = H_n(\Psi(t), S, t)$ be the option value of VA or EIA and $\Psi(t)$ be the survival probability at time t. Intuitively, we have

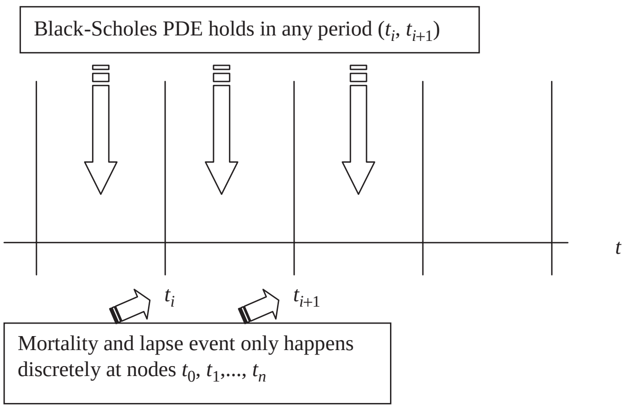
$$H_n(\Psi(t), S, t) = \Psi(t) \cdot H_n(1, S, t)$$

This implies that option value H_n is proportional to survival probability $\Psi(t)$. For example, the price of an option with half policyholders left should be exactly half of the price when 100% of policyholders stay in the contract, given other conditions unchanged. Therefore, we let notation $H_n(S, t)$ stand for $H_n(1, S, t)$ for simplicity. Similarly, we have

$$f_n(\Psi(t), S, t) = \Psi(t) \cdot f_n(1, S, t)$$

In addition, we use $f_n(S, t)$ for $f_n(1, S, t)$. We call $H_n(S, t)$ and $f_n(S, t)$ as all-survival prices. Both $H_n(S, t)$ and $f_n(S, t)$ are Markovians and can be solved through a partial differential equation (PDE) approach.

In addition, we introduce the discrete mortality and lapse model (DMLM): Assuming mortality, lapse behavior and fee charging only happen discretely at nodes $t_0, t_1, \ldots, t_N$, as shown in the following graph:



The Black–Scholes PDE holds for both $H_n(S, t)$ and $f_n(S, t)$ in between every open interval (t_i, t_{i+1}), because no mortality, lapse, or fee charge event happens between nodes:

$$\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rS\frac{\partial V}{\partial S} + rV = 0 \quad \text{on} \ \ (t_i, t_{i+1})$$

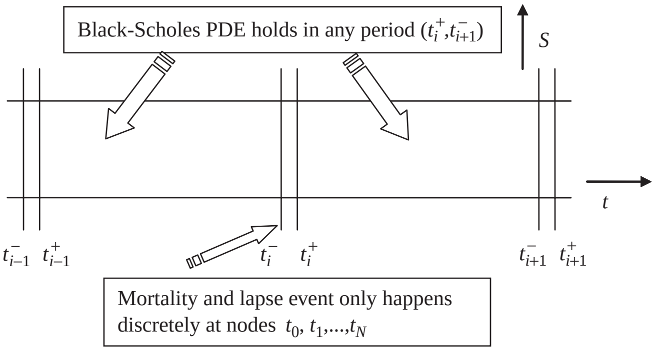
In this PDE, t stands for time to maturity. Let $x = \ln(S)$ to get constant coefficients.

$$\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x^2} - \left(r - \delta - \frac{1}{2}\sigma^2\right)\frac{\partial V}{\partial x} + rV = 0 \quad \text{on } (t_i, t_{i+1}) \tag{2}$$

Under the DMLM framework, the finite difference method will be applicable.

2.3.2. Crank–Nicholson Scheme

For any time node t_i, we split it into two nodes t_i^- and t_i^+, which are infinitely close.[11]



We use the Crank–Nicholson scheme here for its second-order accuracy and non-conditional convergence. Let $V_k^n = V_k^n(S_k, t_n^+)$ and $V_k^{n+1} = V_k^{n+1}(S_k, t_{n+1}^-)$; the discretization form of Eq. (2) is

$$\begin{aligned}\frac{V_k^{n+1} - V_k^n}{\Delta t} = {} & \frac{1}{2}\left[\frac{1}{2}\sigma^2 \frac{V_{k+1}^n - 2V_k^n + V_{k-1}^n}{\Delta x^2} + \left(r - \delta - \frac{1}{2}\sigma^2\right)\frac{V_{k+1}^n - V_{k-1}^n}{2\Delta x} - rV_k^n\right] \\ & + \frac{1}{2}\left[\frac{1}{2}\sigma^2 \frac{V_{k+1}^{n+1} - 2V_k^{n+1} + V_{k-1}^{n+1}}{\Delta x^2} + \left(r - \delta - \frac{1}{2}\sigma^2\right)\frac{V_{k+1}^{n+1} - V_{k-1}^{n+1}}{2\Delta x} - rV_k^{n+1}\right]\end{aligned}$$

This can be simplified in matrix form:

$$\left(\mathbf{I} - \frac{1}{2}\Delta t\mathbf{A}\right) \cdot V^{n+1} = \left(\mathbf{I} + \frac{1}{2}\Delta t\mathbf{A}\right) \cdot V^n$$

where **I** is identity matrix and **A** is triangular with constant coefficients. Eq. (2) has accuracy of $O(\Delta x^2 + \Delta t^2)$ and can be solved quickly by Thomas' algorithm with FLOP counts $O(N)$.

Special attention pays to all time nodes $t_n^{\pm}$ where mortality, lapse, and fee charge occur. According to DMLM, at $t_n^{\pm}$, there is $q\Delta t$ percentage of policyholders die (implying that portion of the total option is exercised) and

$l\Delta t$ percentage of policyholders lapse (meaning that portion of option is abandoned). This can be incorporated into schemes as follows:

$$\text{VA}: H_n(S, t^+) = q\Delta t \max(K - S, 0) + (1 - q\Delta t - l\Delta t)H_n(S, t^-)$$

$$\text{EIA}: H_n(S, t^+) = q\Delta t \max(S - K, 0) + (1 - q\Delta t - l\Delta t)H_n(S, t^-)$$

Both q and l depend on (S, t) and are computed at each grid (S_k, t_n).

For fee charge f_n, it satisfies PDE (Eq. (2)) inside each interval (t_i^+, t_{i+1}^-). At time node $t_n^{\pm}$, according to DMLM, $q\Delta t + l\Delta t$ units of policyholders exit their contract; meanwhile, there are $mS\Delta t$ (for VA) or $(r - r_g)\eta\Delta t$ (for EIA) amount of extra fee charged. This can be modeled as follows:

$$\text{VA}: f_n(S, t^+) = (1 - q\Delta t - l\Delta t) \cdot (f_n(S, t^-) + Sm\Delta t)$$

$$\text{EIA}: f_n(S, t^+) = (1 - q\Delta t - l\Delta t) \cdot (f_n(S, t^-) + (r - r_g)\eta\Delta t)$$

The initial condition of VA and EIA option value $H_n(S, t)$ is contract's payoff function at maturity $t = 0$. The fee charge f_n has zero initial value for both VA and EIA:

$$\text{VA}: H_n(S, 0) = \max(K - S, 0)$$

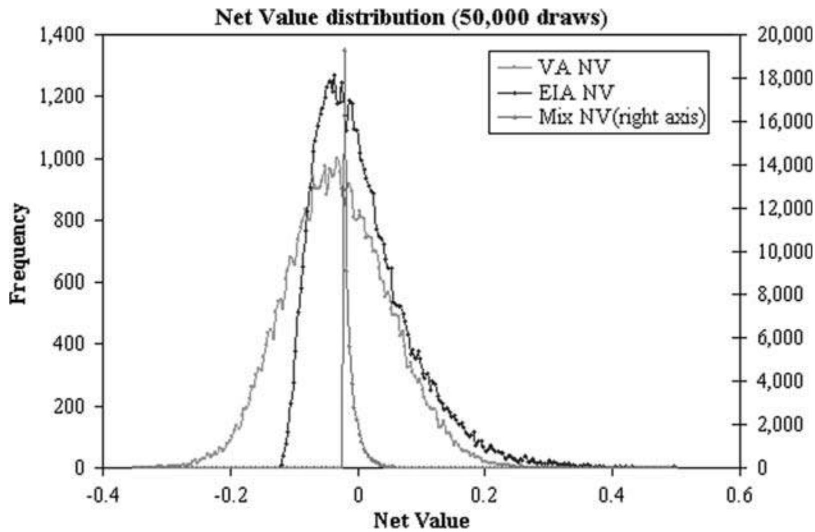
$$\text{EIA}: H_n(S, 0) = \max(S - K, 0)$$

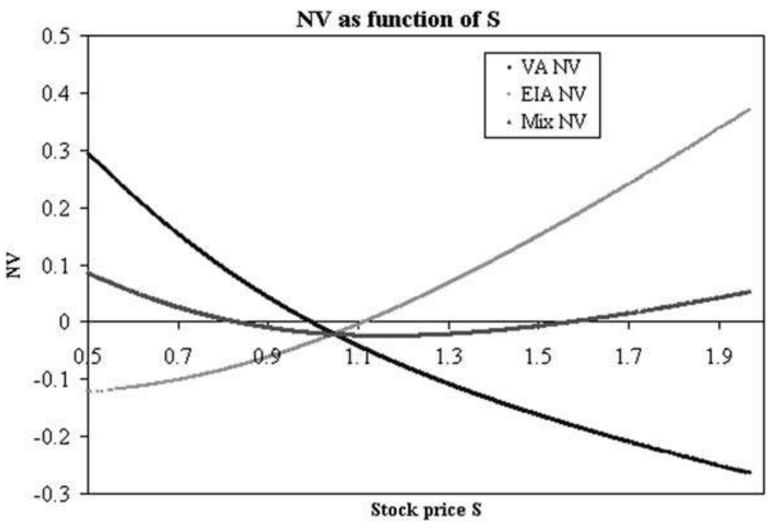
$$f_n(S, 0) = 0$$

2.4. Economic Capital and Conditional VaR for VA and EIA Mixture

As introduced at the beginning of this chapter, natural diversification effects exist for a portfolio that includes both VA (which is put-like) and EIA (which is call-like) products. Suppose both products share the same underlying equity process. Then, such a portfolio can be modeled as a *straddle* (or *strangle*), that is, whenever either product is in-the-money, the other one is likely to be out-of-the-money. More specifically, when stock price is low and VA is in-the-money, the option value embedded in EIA drops and draws the portfolio value to remain regular; when the stock price is high and EIA is in-the-money, not only is the option value embedded in VA drop, but also the policyholder's account was charged by the insurance company with higher management fees; both lower the total loss of the whole portfolio. Therefore, the risk to the insurer that provides these products is reduced. In this chapter, we pick both VaR and Conditional VaR[12] (CVaR) as risk measures.

The following figures illustrate the diversification effect between VA and EIA. The first figure is the price distribution histogram of VA, EIA, and a mixture that contains 50% VA and 50% EIA. Compared to VA or EIA, the mixture has a very concentrated distribution range around zero. The second figure is the price of VA, EIA, and mixture as a function of underlying stock price. Compared to VA or EIA, the mixture curve is flatter and less sensitive to stock moves. These all imply a smaller VaR.





In the next section, we will introduce the Monte Carlo simulation/finite difference hybrid framework to calculate EC.

2.4.1. Economic Capital and Conditional VaR Calculation

In this chapter, a Monte Carlo simulation/finite difference hybrid framework is used to valuate EC of both products and the diversification benefits. The simulation algorithm consists of the following steps:

1. Divide time t and space $x = \ln(S)$ up into M by N discrete intervals:

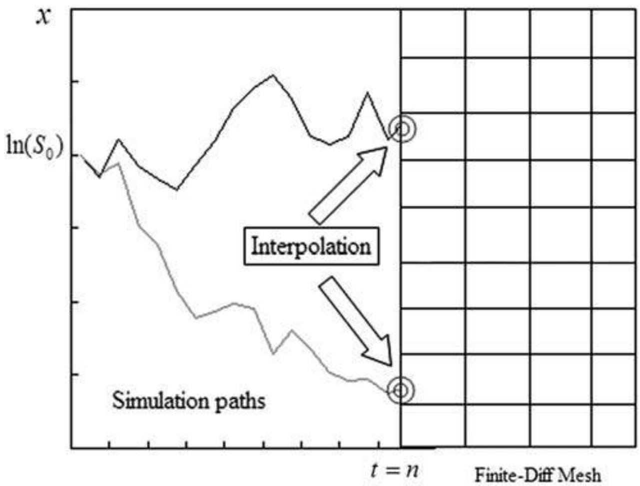
$$0 = t^1 < t^2 < \ldots < t^M = T$$

$$X_{\text{Min}} = x^1 < x^2 < \ldots < x^N = X_{\text{Max}}$$

   Let $\{S^k = \exp(x^k) | k = 1, \ldots, M\}$ be the stock space nodes.
2. Set up the finite difference grid. Given any time horizon n, we solve PDE backwards till time $t = n$. We can get option and fee all-survival prices at all space nodes at $t = n$: $\{H_n^k | k = 1, \ldots, N\}$ and $\{f_n^k | k = 1, \ldots, N\}$.
3. Simulate the equity price paths from time 0 to time $t = n$. Let NSM be the total number of simulations runs. The equity price at $t = n$ is $\{S_{n,i} | i = 1, \ldots, \text{NSM}\}$. We also simulate survival probability $\Psi(t)$ along these equity paths till $t = n$ (taking mortality $q(t)$ and lapse rate $l(S, t)$ into account): $\{\Psi_{n,i} | i = 1, \ldots, \text{NSM}\}$. This is called the outer simulation paths.
4. At time $t = n$ for each simulation path i, we have equity price $S_{n,i}$. Supposing $S_{n,i}$ is located between two adjoining stock space nodes, $S^{\tilde{k}} < S_{n,i} < S^{\tilde{k}+1} < S^{\tilde{k}+2}$, we interpolate a quadratic polynomial[13] between $\{H_n^k, f_n^k | k = \tilde{k}, \tilde{k}+1, \tilde{k}+2\}$ to compute all-survival option value $\tilde{H}_{n,i}$ and fee $\tilde{f}_{n,i}$, respectively.
5. The actual option, fee, and VA/EIA net value are calculated as follows:

$$H_{n,i} = \Psi_{n,i} \cdot \tilde{H}_{n,i}$$
$$f_{n,i} = \Psi_{n,i} \cdot \tilde{f}_{n,i}$$
$$\text{NV}_{n,i} = H_{n,i} - f_{n,i}$$

6. Repeat steps 4 and 5 until we get VA and EIA net value $\text{NV}_{n,i}$ for all simulation paths $i = 1, \ldots, \text{NSM}$ as shown in the figure below. We can compute EC and Conditional VaR at 99% level of $\text{NV}_{n,i}$.



For a portfolio P that includes both VA and EIA products, let w be the weight of VA. We can optimize w to minimize the portfolio's EC[14] at any time horizon $t = n$:

$$\min_{0<w<1}(\text{VaR}[w\text{NV}_n^{\text{VA}} + (1-w)\text{NV}_n^{\text{EIA}}])$$

w can be optimized through usual iteration algorithms such as Newton's method or gradient descent. The following graph is an example of the portfolio EC as the function of w:

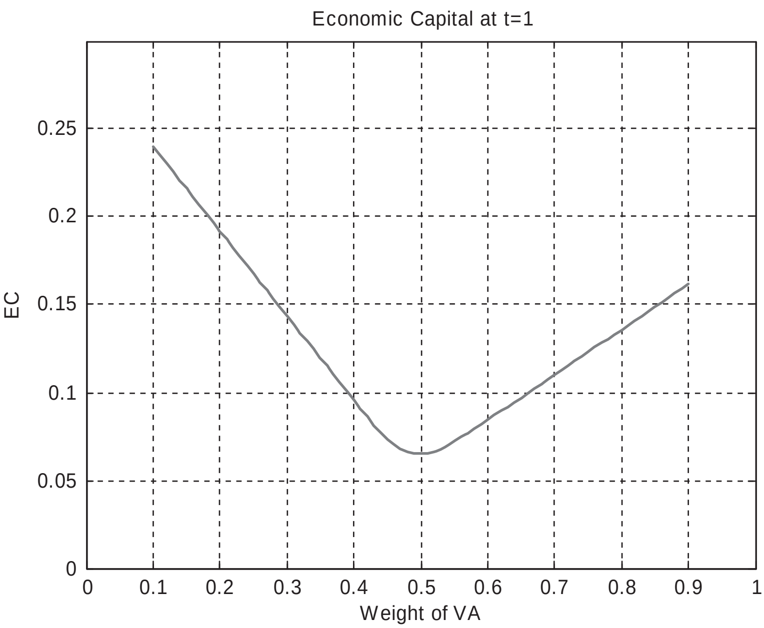


Table 2 provides the EC requirements and conditional VaR for VA, EIA, and the optimal VA/EIA mixture based on different time horizons.[15] The optimal weight column is the percentage weight of VA in the optimal portfolio. Graphical results are listed in Fig. 4.

In this example, the natural hedging effect is significant. In the first 5 years, the optimal mixtures have an average of 43% smaller EC requirements than VA, and these are 78% smaller compared to EIA. These optimal mixtures also have Conditional VaR that are superior to any single product.

Table 2. Economic Capital Requirements and CVaR for VA, EIA, and Mixture.

Tenor (Years)	Economic Capital (VaR 99%)			Optimal Weight (VA, %)	Conditional VaR		
	VA	EIA	Optimal portfolio		VA	EIA	Optimal portfolio
1	0.19	0.29	***0.07***	50	0.22	0.35	***0.08***
2	0.26	0.57	***0.13***	57	0.31	0.70	***0.16***
3	0.32	0.89	***0.19***	64	0.37	1.11	***0.23***
4	0.37	1.26	***0.25***	70	0.43	1.57	***0.30***
5	0.42	1.70	***0.31***	75	0.48	2.13	***0.36***

Note: The bold/italic numbers are used to show the effect of risk reduction (in terms of Eco Capital and Cond. VaR) from an optimized VA/EIA portfolio.

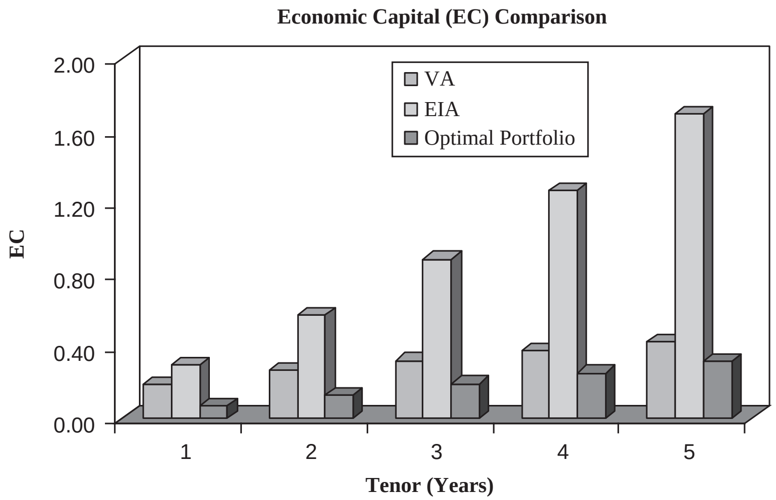


Fig. 4. Economic Capital Requirement for VA, EIA, and the Optimal Mixture.

It is also observed in Table 2 that the EC of EIA goes up tremendously. This is because for shortening a call, there is not an upper bound for the future loss. While in the mixture portfolio, the loss from EIA is balanced out by the moneyness of the option embedded in VA and the fees charged from policyholder's account, as shown in the optimal portfolio column. In Table 2, the weights of VA in the optimal mixture portfolio grow gradually. This is because the capital demand from EIA increases quickly and thus requires more weight in the VA to offset.

2.5. Conclusion

This chapter contributes to the literature in the area of equity-linked insurance contract pricing and analyzing natural diversification benefits between VA and EIA products. These benefits result from the reason that the values of VA and EIA move in opposite directions in response to a change in the underlying equity value. The author modeled VA and EIA in the risk-neutral option pricing framework and implemented finite difference pricing scheme. Numerical examples show that natural hedging is feasible and the benefits are significant, which enables insurance companies' capital to be deployed more efficiently.

NOTES

1. *Source*: National Association for Variable Annuities (NAVA).
2. The term "lapse" means a policyholder unwinds his insurance contract, liquidates his account, and exits. Usually a certain penalty fee is necessary for the cost of breaking an existing contract.
3. Economic capital (EC) in this chapter is defined as the difference between 99% Value at Risk (VaR) of product's net value and initial value: $\mathrm{EC}(t) = \mathrm{VaR}_{99\%}(V(t)) - V(0)$.
4. Monotonicity of function $\mathrm{NV}_n(S_n)$ is implied by the negativeness of its first derivative with respect to S_n.
5. For incomplete mortality market analysis, please refer to Follmer and Sonderman (1986).
6. This is because life insurance policyholders are neither financial professionals nor institutional investors, and lapse does happen for reasons unrelated to the equity performance. Liquidity problems and defaults can be examples.
7. Base lapse rate can be influenced by macro-economic factors such as domestic economy and federal rates.
8. Differs from VA, the owners of general accounts could lose part or all of their investments if the insurer defaults.

9. Monotonicity of function $NV_n(S_n)$ is implied by the positiveness of H_n's first derivative with respect to S_n. Here f_n is not a function of S_n and therefore has no contribution to $d(NV_n(S_n))/dS_n$.

10. Monte Carlo simulation results can be greatly improved if we take the no-mortality/lapse option value as a control variate. However, such approach would still be a lot slower than using the finite difference algorithm described in this chapter.

11. The split technique here is for implementation purpose only.

12. Conditional VaR is defined as the conditional expectation of random variable that exceeds its VaR: $\text{CVaR}_\alpha(X) = E(X|X > \text{VaR}_\alpha(X))$. CVaR is *coherent* and therefore is usually considered as a better alternative risk measure to VaR. CVaR is also called Expected Shortfall or Expected Tail Loss in Finance. Please refer to Pflug (2000) for more detail.

13. We use quadratic interpolation here to be consistent with the second-order accuracy of Crank–Nicholson scheme.

14. Alternatively we can run optimization targeting CVaR of the portfolio, which is not covered in the scope of this chapter.

15. Valuation parameters are as follows:

VA: Maturity $N = 10$ years, guaranteed interest rate $r_g = 2\%$, premium charge 2%.

EIA: Maturity $N = 10$ years, guaranteed interest rate $r_g = 2\%$, guaranteed amount $\eta = 100\%$, participation rate $\alpha = 70\%$

Mortality rate: 1%, base lapse rate: 2%. Equity drift $\mu = 12\%$, volatility $\sigma = 0.2$, dividend yield $d = 2\%$.

16. Here, the authors intentionally skipped rigorous mathematical proof of the interchange of derivative and integral. Precisely, this formula is valid only when the following technical conditions hold: (1) Both $\Psi(t)q(t)\text{BSP}(n, t)$ and $d(\Psi(t)q(t)\text{BSP}(n, t))/dS_n$ are continuous; (2) Both $\Psi(t)q(t)\text{BSP}(n, t)$ and $d(\Psi(t)q(t)\text{BSP}(n, t))/dS_n$ are bounded by a L^1 function. See Cheney (2001) for example.

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APPENDIX

Notations

- G_n ▸ guaranteed level
- r_g ▸ guaranteed interest rate
- N ▸ maturity of product
- F_n ▸ account value at time n
- H_n ▸ value of embedded option

S_0	▸ underlying equity price at time 0
m	▸ management fee of VA charged each year
η	▸ guaranteed amount of EIA
α	▸ participation rate
$V_{\text{put}}(S_0, K, r, d, \sigma, t)$	▸ price of a vanilla European put
$\Psi(t)$	▸ survival probability
$l(S_t, t)$	▸ instantaneous lapse rate
$q(t)$	▸ instantaneous mortality rate
NV_n	▸ net value of guarantee
f_n	▸ value of benefit charge
β	▸ confidence level
EC	▸ economic capital
CVaR	▸ Conditional Value at Risk
$BSC(n, t)$	▸ European call price at time n and maturities at t
$\text{BSP}(n, t)$	▸ European put price at time n and maturities at t

Proof of Proposition 1

Proposition 1. In the case where both mortality and lapse risk are independent from the underlying equity price, the function $\text{NV}_n(S_n)$ is monotonically decreasing.

Proof. If both risks are independent from the underlying equity price S_n, $\text{NV}_n(S_n)$ has the following form:

$$\text{NV}_n(S_n) = H_n - f_n$$

with

$$H_n = \int_n^N \Psi(t)q(t)\text{BSP}(n, t)dt + \Psi(N) \cdot \text{BSP}(n, N)$$

The PV of the fees is

$$f_n = \frac{\varepsilon S_n}{S_0} \int_n^N \mathrm{e}^{-\int_0^t [l(u)+q(u)]du} \cdot \mathrm{e}^{-m(t-n)} dt$$

Now, take the first derivative of both H_n and f_n with respect to S_n:[16]

$$\frac{dH_n}{dS_n} = \int_n^N \Psi(t)q(t)\frac{d(\text{BSP}(n, t))}{dS_n} dt + \Psi(N) \cdot \frac{d(\text{BSP}(n, N))}{dS_n}$$

As

$$\frac{d(\mathrm{BSP}(n,t))}{dS_n} = -\frac{\mathrm{e}^{-mt}}{S_0}\mathrm{e}^{-d(t-n)}\Phi(-d_1) < 0$$

we know that H_n is monotonically decreasing. For f_n,

$$\frac{df_n}{dS_n} = \frac{\varepsilon}{S_0}\int_n^N \mathrm{e}^{-\int_0^t [l(u)+q(u)]du} \cdot \mathrm{e}^{-m(t-n)} dt \geq 0$$

which implies that f_n is monotonically increasing. Therefore, $\mathrm{NV}_n(S_n)$ is monotonically decreasing. ■